After reading this lesson, you will learn about

- Fourier series expansion (Trigonometric and exponential)
- Properties of Fourier Series
- Response of a linear system
- Normalized power in a Fourier expansion
- Power spectral density
- Effect of transfer function on PSD

1. FOURIER SERIES EXPANSION:

French Mathematician J.B.J. Fourier found that any arbitrary periodic signal can be represented with an infinite series of sinusoids with fundamental frequency and harmonically related frequency ($\omega_0 =$fundamental frequency, $n \omega_0 =$ n$^{th}$ harmonic frequency, where n=1, 2, 3, 4… N). Fourier analysis is used to analysis the steady state response of a network and frequency analysis of signals.

Periodic Function:

A function is said to be periodic with a time period ‘T’ if it satisfies the relation $f(t \pm T) = f(t)$. A numbers of such periodic signals are shown in the fig. below. Thus a periodic function repeats itself after every T seconds.

![Fig.1](image1.png)

There are two forms of Fourier series

(i) Trigonometric Fourier series
(ii) Exponential Fourier Series
Trigonometric Fourier series:

The trigonometric Fourier series for an arbitrary periodic function \( f(t) \) is given by

\[
f(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \quad (1)
\]

Where \( a_n \)'s and \( b_n \)'s are known as the Fourier series coefficients.

Key roots (Formulae)

\[
\int_0^T \sin n\omega_0 t \, dt = 0 \text{ for all } 'n'
\]

\[
\int_0^T \cos m\omega_0 t \, dt = 0, \text{ when } m \neq 0
\]

\[
\int_0^T \sin n\omega_0 t \cdot \cos m\omega_0 t \, dt = 0 \text{ for all } m, n
\]

\[
\int_0^T \sin n\omega_0 t \cdot \sin m\omega_0 t \, dt = 0 \text{ for } m \neq n
\]

\[
\int_0^T \cos n\omega_0 t \cdot \cos m\omega_0 t \, dt = 0 \text{ for } m \neq n
\]

\[
\int_0^T \sin^2 n\omega_0 t \, dt = \frac{T}{2} \text{ for } n \neq 0
\]

\[
\int_0^T \cos^2 n\omega_0 t \, dt = \frac{T}{2}
\]

Evaluation of Fourier series co-efficient:

This involves two operations

(i) Evaluation of coefficients \( a_0, a_n \) and \( b_n \)

(ii) Truncating of infinite series after a finite number of terms so that \( f(t) \) is represented within allowable error.
**Evaluation of** $a_0$:

\[ f(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right] \]

Integrating both sides within one time period

\[
\int_{0}^{T} f(t) \, dt = \int_{0}^{T} a_0 \, dt + \int_{0}^{T} \left\{ \sum_{n=1}^{\infty} \left[ a_n \cos n \omega_0 t + b_n \sin n \omega_0 t \right] \right\} \, dt
\]

NB: Integration of a sinusoidal function within limit 0-T, (where T is the time period of the given function) is zero.

\[
\Rightarrow \int_{0}^{T} f(t) \, dt = a_0 [T] + 0
\]

\[
\Rightarrow \int_{0}^{T} f(t) \, dt = a_0 [T - 0]
\]

\[
\Rightarrow a_0 = \frac{1}{T} \int_{0}^{T} f(t) \, dt \quad \text{........(2)}
\]

$a_0$ is known as the average value of the function or D.C. value of the function.

**Evaluation of** $a_n$

Multiplying $\cos n\omega_0 t$ in Equation (1) and integrating both sides over a period 0-T, we will get

\[
\int_{0}^{T} f(t) \cos n\omega_0 t \, dt = \int_{0}^{T} a_0 \cos n\omega_0 t \, dt + \int_{0}^{T} \left\{ \cos n\omega_0 t \left( \sum_{n=1}^{\infty} \left[ a_n \cos (n \omega_0 t) + b_n \sin (n \omega_0 t) \right] \right\} \rightdt
\]

\[
\Rightarrow \int_{0}^{T} f(t) \cos n\omega_0 t \, dt = \int_{0}^{T} a_0 \cos n\omega_0 t \, dt + \sum_{n=1}^{\infty} \int_{0}^{T} a_n \cos n\omega_0 t \cdot \cos n\omega_0 t \, dt + \sum_{n=1}^{\infty} \int_{0}^{T} b_n \sin n\omega_0 t \cdot \cos n\omega_0 t \, dt
\]

\[
\Rightarrow \int_{0}^{T} f(t) \cos n\omega_0 t \, dt = 0 + a_n (\frac{T}{2}) + 0
\]

\[
\Rightarrow a_n = \frac{2}{T} \int_{0}^{T} f(t) \cos n\omega_0 t \, dt \quad \text{........(3)}
\]

**Evaluation of** $b_n$

Multiplying $\sin n\omega_0 t$ in Equation (1) and integrating both sides over a period 0-T, we will get
\[ \int_{0}^{T} f(t) \sin n \omega_0 t \, dt = \int_{0}^{T} a_0 \sin n \omega_0 t \, dt + \sum_{n=1}^{\infty} \left[ a_n \cos n \omega_0 t + b_n \sin n \omega_0 t \right] \]

\[ \Rightarrow \int_{0}^{T} f(t) \sin n \omega_0 t \, dt = \int_{0}^{T} a_0 \sin n \omega_0 t \, dt + \sum_{n=1}^{\infty} a_n \cos n \omega_0 t \cdot \sin n \omega_0 t \, dt + \sum_{n=1}^{\infty} b_n \sin n \omega_0 t \cdot \sin n \omega_0 t \, dt \]

\[ \Rightarrow \int_{0}^{T} f(t) \sin n \omega_0 t \, dt = 0 + b_n \left( \frac{T}{2} \right) + 0 \]

\[ \Rightarrow b_n = \frac{2}{T} \int_{0}^{T} f(t) \sin n \omega_0 t \, dt \quad (4) \]

**Types of Symmetry:**

(i) Even symmetry

(ii) Odd symmetry

(iii) Half wave symmetry

(i) **Even symmetry:** If a function satisfies the relation \( f(t) = f(-t) \) it is said to be even symmetry.

Examples:

- \( \cos \theta \) is an even symmetry function

(ii) **Odd symmetry:** If a function satisfies the relation \( f(t) = -f(-t) \) then it is an odd symmetry function.
Examples:

(iii) **Half wave symmetry**: If a function satisfies the relation $f(t) = -f(t + T/2)$, then it is a half wave symmetry function.

Example:

![Example of half wave symmetry](image1)

The amplitude is taken as $A$.

Take $t = T/4$

$f(t) = f(T/4) = A$

$f(T/4 + T/2) = f(3T/4) = -A$

Similarly

$f(t - T/2) = f(-T/4) = -A$

So, Sine functions are half wave symmetry.

**Evaluation of Fourier Coefficients in Symmetry Conditions**

(i) **Even symmetry**: 

**Evaluation of $a_0$**: 

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{1}{T} \int_{-T/2}^0 f(t) dt + \frac{1}{T} \int_0^{T/2} f(t) dt$$

Replace $t$ by $-t$ in the first term.

$$\Rightarrow a_0 = \frac{1}{T} \int_{-T/2}^{0} f(-t)(-dt) + \frac{1}{T} \int_0^{T/2} f(t) dt$$

$f(t)$ is even symmetry function, so $f(t) = f(-t)$.
\[ a_0 = \frac{1}{T} \int_0^{T/2} f(t)dt + \frac{1}{T} \int_{-T/2}^0 f(t)dt = \frac{2}{T} \int_0^{T/2} f(t)dt \]  \hspace{1cm} (5)

**Evaluation of \( a_n \):**

\[ a_n = \frac{2}{T} \int_0^T f(t) \cos nw_0 t dt = \frac{2}{T} \int_0^{T/2} f(t) \cos nw_0 t dt \]

\[ = \frac{2}{T} \int_0^{T/2} f(t) \cos nw_0 t dt + \frac{2}{T} \int_{-T/2}^0 f(t) \cos nw_0 t dt \]

Replace \( t \) with \( -t \) in the first term

\[ = \frac{2}{T} \int_0^{T/2} f(-t) \cos(-nw_0 t)(-dt) + \frac{2}{T} \int_0^{T/2} f(t) \cos nw_0 t dt \]

\( f(t) \) is even symmetry function, so \( f(t) = f(-t) \)

\[ = \frac{2}{T} \int_0^{T/2} f(t) \cos(nw_0 t)(dt) + \frac{2}{T} \int_0^{T/2} f(t) \cos nw_0 t dt \]

\[ = \frac{4}{T} \int_0^{T/2} f(t) \cos nw_0 t dt \]  \hspace{1cm} (6)

**Evaluation of \( b_n \):**

\[ b_n = \frac{2}{T} \int_0^T f(t) \sin nw_0 t dt = \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t dt \]

\[ = \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t dt + \frac{2}{T} \int_{-T/2}^0 f(t) \sin nw_0 t dt \]

Replace \( t \) with \( -t \) in the first term

\[ = \frac{2}{T} \int_0^{T/2} f(-t) \sin(-nw_0 t)(-dt) + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t dt \]

\( f(t) \) is even symmetry function, so \( f(t) = f(-t) \)

\[ = -\frac{2}{T} \int_0^{T/2} f(t) \sin(nw_0 t)(dt) + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t dt \]

\[ = 0 \]  \hspace{1cm} (7)

(ii) **Odd symmetry:**

**Evaluation of \( a_0 \):**

\[ a_0 = \frac{1}{T} \int_0^T f(t)dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t)dt = \frac{1}{T} \int_{-T/2}^0 f(t)dt + \frac{1}{T} \int_0^{T/2} f(t)dt \]

Replace \( t \) by \(-t\) in the first term
\[ a_0 = \frac{1}{T} \int_{0}^{T/2} f(-t)(-dt) + \frac{1}{T} \int_{0}^{T/2} f(t)dt \]

\( f(t) \) is odd symmetry function, so \( f(t) = -f(-t) \)

\[ a_0 = -\frac{1}{T} \int_{0}^{T/2} f(t)dt + \frac{1}{T} \int_{0}^{T/2} f(t)dt = 0 \]  \( \text{(8)} \)

**Evaluation of } a_n:***

\[ a_n = \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega_0 t dt = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t dt \]

\[ = \frac{2}{T} \int_{-T/2}^{0} f(t) \cos n\omega_0 t dt + \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega_0 t dt \]

Replace \( t \) with \(-t\) in the first term

\[ = \frac{2}{T} \int_{0}^{T/2} f(-t) \cos(-n\omega_0 t)dt + \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega_0 t dt \]

\( f(t) \) is odd symmetry function, so \( f(t) = -f(-t) \)

\[ = -\frac{2}{T} \int_{0}^{T/2} f(t) \cos(n\omega_0 t)dt + \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega_0 t dt \]

\[ = 0 \]  \( \text{(9)} \)

**Evaluation of } b_n:**

\[ b_n = \frac{2}{T} \int_{0}^{T/2} f(t) \sin n\omega_0 t dt = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t dt \]

\[ = \frac{2}{T} \int_{-T/2}^{0} f(t) \sin n\omega_0 t dt + \frac{2}{T} \int_{0}^{T/2} f(t) \sin n\omega_0 t dt \]

Replace \( t \) with \(-t\) in the first term

\[ = \frac{2}{T} \int_{0}^{T/2} f(-t) \sin(-n\omega_0 t)(-dt) + \frac{2}{T} \int_{0}^{T/2} f(t) \sin n\omega_0 t dt \]

\( f(t) \) is odd symmetry function, so \( f(t) = -f(-t) \)

\[ = \frac{2}{T} \int_{0}^{T/2} f(t) \sin(n\omega_0 t)dt + \frac{2}{T} \int_{0}^{T/2} f(t) \sin n\omega_0 t dt \]

\[ = \frac{4}{T} \int_{0}^{T/2} f(t) \sin n\omega_0 t dt \]  \( \text{(10)} \)
(iii) Half wave symmetry:

![Graph of half wave symmetry function]

**Evaluation of $a_0$:**

$$a_0 = \frac{1}{T} \int_{0}^{T} f(t) dt = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{1}{T} \int_{0}^{T/2} f(t) dt + \frac{1}{T} \int_{-T/2}^{0} f(t) dt$$

Replace $(t+T/2)$ by $\lambda$ in the first term ; $t=\lambda-T/2$ and $dt=d\lambda$

$$\Rightarrow a_0 = \frac{1}{T} \int_{0}^{T/2} f(\lambda-T/2) d\lambda + \frac{1}{T} \int_{0}^{T/2} f(t) dt$$

$f(t)$ is half wave symmetry function, so $f(\lambda-T/2)=-f(\lambda)$

$$\Rightarrow a_0 = -\frac{1}{T} \int_{0}^{T/2} f(\lambda) d\lambda + \frac{1}{T} \int_{0}^{T/2} f(t) dt = 0 \quad (11)$$

**Evaluation of $a_n$:**

$$a_n = \frac{2}{T} \int_{0}^{T} f(t) \cos (nw_0 t) dt = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos (nw_0 t) dt$$

$$= \frac{2}{T} \int_{0}^{T/2} f(t) \cos (nw_0 t) dt + \frac{2}{T} \int_{-T/2}^{0} f(t) \cos (nw_0 t) dt$$

Replace $(t+T/2)$ by $\lambda$ in the first term ; $t=\lambda-T/2$ and $dt=d\lambda$

$$= \frac{2}{T} \int_{0}^{T/2} f(\lambda-T/2) \cos (nw_0 (\lambda-T/2)) d\lambda + \frac{2}{T} \int_{0}^{T/2} f(t) \cos (nw_0 t) dt$$

$f(t)$ is half wave symmetry function, so $f(\lambda-T/2)=-f(\lambda)$

$$= -\frac{2}{T} \int_{0}^{T/2} f(\lambda) \cos (nw_0 \lambda - nw_0 T/2) d\lambda + \frac{2}{T} \int_{0}^{T/2} f(t) \cos nw_0 t dt$$

$$= -\frac{2}{T} \int_{0}^{T/2} f(\lambda) \cos (nw_0 \lambda - n\pi) d\lambda + \frac{2}{T} \int_{0}^{T/2} f(t) \cos nw_0 t dt$$

$$\cos(nw_0 \lambda - n\pi) = \begin{cases} \cos nw_0 \lambda & \text{when } n = \text{even} \\ -\cos nw_0 \lambda & \text{when } n = \text{odd} \end{cases}$$
So \( a_n = \begin{cases} 
0 & \text{for } n = \text{even} \\
\frac{4}{T} \int_0^{T/2} f(t) \cos nw_0 t \, dt & \text{for } n = \text{odd} 
\end{cases} \quad (12) 
\]

**Evaluation of \( b_n \):**

\[
b_n = \frac{2}{T} \int_0^T f(t) \sin nw_0 t \, dt = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin nw_0 t \, dt 
\]

\[
= \frac{2}{T} \int_{-T/2}^0 f(t) \sin nw_0 t \, dt + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t \, dt 
\]

Replace \((t+T/2)\) by \(\lambda\) in the first term; \(t=\lambda - T/2\) and \(dt=d\lambda\)

\[
= \frac{2}{T} \int_0^{T/2} f(\lambda - T/2) \sin(nw_0 (\lambda - T/2)) \, d\lambda + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t \, dt 
\]

\(f(t)\) is half wave symmetry function, so \(f(\lambda - T/2) = -f(\lambda)\)

\[
= -\frac{2}{T} \int_0^{T/2} f(\lambda) \sin(nw_0 \lambda - nw_0 T/2) \, d\lambda + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t \, dt 
\]

\[
= -\frac{2}{T} \int_0^{T/2} f(\lambda) \sin(nw_0 \lambda - n\pi) \, d\lambda + \frac{2}{T} \int_0^{T/2} f(t) \sin nw_0 t \, dt 
\]

\[
\sin(nw_0 \lambda - n\pi) = \begin{cases} 
\sin nw_0 \lambda & \text{when } n = \text{even} \\
-\sin nw_0 \lambda & \text{when } n = \text{odd} 
\end{cases} 
\]

So \( b_n = \begin{cases} 
0 & \text{for } n = \text{even} \\
\frac{4}{T} \int_0^{T/2} f(t) \sin nw_0 t \, dt & \text{for } n = \text{odd} 
\end{cases} \quad (13) 
\]

The Fourier series expansion for a function having half wave symmetry contains only the odd harmonics frequencies.

**SUMMARY:**

(i) **Even Symmetry**

\[
a_0 = \frac{2}{T} \int_0^T f(t) \, dt 
\]

\[
a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos nw_0 t \, dt 
\]

\(b_n = 0\)

(ii) **Odd Symmetry**

\[
a_0 = 0 
\]
\( a_n = 0 \)
\( b_n = \frac{4}{T} \int_{0}^{T/2} f(t) \sin nw_0 t \, dt \)

(iii) Half wave symmetry
\( a_0 = 0 \)
\( a_n = 0, \quad b_n = 0 \quad \text{for } n = \text{even} \)
\( a_n = \frac{4}{T} \int_{0}^{T/2} f(t) \cos nw_0 t \, dt \quad \text{for } n = \text{odd} \)
\( b_n = \frac{4}{T} \int_{0}^{T/2} f(t) \sin nw_0 t \, dt \quad \text{for } n = \text{odd} \)

If any function contains
(i) Even symmetry and half wave symmetry
\( a_0 = 0, \quad b_n = 0 \)
\( a_n = 0 \quad \text{for } n = \text{even} \)
\( a_n = \frac{4}{T} \int_{0}^{T/2} f(t) \cos nw_0 t \, dt \quad \text{for } n = \text{odd} \)
\( = \frac{8}{T} \int_{0}^{T/4} f(t) \cos nw_0 t \, dt \quad \text{for } n = \text{odd} \)

(ii) Odd symmetry and half wave symmetry
\( a_0 = 0, \quad a_n = 0 \)
\( b_n = 0 \quad \text{for } n = \text{even} \)
\( b_n = \frac{4}{T} \int_{0}^{T/2} f(t) \sin nw_0 t \, dt \quad \text{for } n = \text{odd} \)
\( = \frac{8}{T} \int_{0}^{T/4} f(t) \sin nw_0 t \, dt \quad \text{for } n = \text{odd} \)

**Exponential Fourier series:**

Let \( v(t) \) is a periodic signal then according to Fourier series
\[
v(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos nw_0 t + b_n \sin nw_0 t \right]
\]
\[ v(t) = v_0 + \sum_{n=-\infty}^{\infty} \left[ v_n e^{j\omega_0 n t} + v_{-n} e^{-j\omega_0 n t} \right] = \sum_{n=-\infty}^{\infty} v_n e^{j\omega_0 n t} \]

Where \( v_n \) is the exponential Fourier series Coefficients.

\[ v_n = \left( \frac{a_n - jb_n}{2} \right) = \frac{1}{2} \left( \frac{2}{T} \int_{0}^{T} f(t) \cos(\omega_0 n t) dt - j \frac{2}{T} \int_{0}^{T} f(t) \sin(\omega_0 n t) dt \right) \]

\[ = \frac{1}{T} \int_{0}^{T} f(t) \cos(\omega_0 n t) dt \]

\[ = \frac{1}{T} \int_{0}^{T} f(t) e^{-j\omega_0 n t} dt \]

We can represent \( v(t) \) as

\[ v(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left( \frac{2\pi n t}{T_0} - \phi_n \right) \]

Where \( C_0, C_n \) and \( \phi_n \) are related to \( a_0, a_n, b_n \) as

\[ C_0 = a_0 \]

\[ C_n = \sqrt{a_n^2 + b_n^2} \]

\[ \phi_n = \tan^{-1} \left( \frac{b_n}{a_n} \right) \]

\( C_n \) is also known as spectral amplitude i.e \( C_n \) is the amplitude of the spectral component \( C_n \cos(2\pi nf_0 t - \phi_n) \) at frequency \( nf_0 \).

**Fourier series Frequency Spectrum:** The plot of the amplitude of frequency component vs the frequency known as the discrete frequency spectrum or line spectrum. The frequency spectrum consists of discrete lines. The length of the line represents the amplitude of the corresponding frequency component.

**Phase Spectrum:** The plot of the phase of the frequency component vs the frequency is known as phase spectrum. Phase spectrum is a odd function.
Sampling Function: The sampling function is defined as $Sa(x) \equiv \frac{\sin x}{x}$. The function is shown in the following figure.

2. FOURIER SERIES PROPERTIES:

If the Fourier series representation of $v(t)$ given as $v(t) = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi n T_0 t}$ then the following properties are satisfied by the signal

(i) **Time shift:** Fourier series representation of $v(t+\tau)$, is

$v(t+\tau) = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi n (t+\tau) / T_0} = \sum_{n=-\infty}^{\infty} v_n' e^{j2\pi n T_0 t}$

Where $v_n' = v_n e^{j2\pi \tau / T_0}$

(ii) **Time inversion:** Fourier series representation of $v(-t)$ is given by

$v(-t) = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi (-t) / T_0} = \sum_{n=-\infty}^{\infty} v_n e^{-j2\pi n T_0 t}$

i.e. the magnitude of $v_n$ remains constant, phase is shifted by 180°. In trigonometric representation $a_n$ remains constant but $b_n$ becomes negative.

(iii) **Time scaling:**

$v(at) = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi n (at) / T_0} = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi n T_0' t}$

Where $T_0' = T_0/a$. i.e. $v_n$ remains constant but shifts to a new frequency $na/T_0$. If the signal is compressed in time domain ($a>1$) it is expanded in frequency domain; and if it is expanded in time domain ($a<1$) then compressed in frequency domain.

(iv) **Time derivative:**

$$\frac{dv(t)}{dt} = \sum_{n=-\infty}^{\infty} \frac{d(v_n e^{j2\pi n T_0 t})}{dt} = \sum_{n=-\infty}^{\infty} (j2\pi n T_0) v_n e^{j2\pi n T_0 t} = \sum_{n=-\infty}^{\infty} v_n' e^{j2\pi n T_0 t}$$
Where $v_n' = j 2\pi n v_n / T_0$

integration:

$$\int_0^t v(t) dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} v_n' e^{j2\pi n t / T_0} dt = \sum_{n=-\infty}^{\infty} v_n' e^{j2\pi n t / T_0} = \sum_{n=-\infty}^{\infty} v_n e^{j2\pi n t / T_0}$$

where $v_n' = v_n / (j 2\pi n / T_0)$

3. APPLICATIONS OF FOURIER SERIES EXPANSION:

(i) Response of a Linear System: When a sinusoidal excitation is applied to a linear system the response of the system is similarly sinusoidal, i.e., sinusoidal waveform preserves the wave shape. The relationship of the response to the excitation is characterized by the relation of input - output amplitude and phase. Let the input to the linear system be the spectral component

$$v_i(t, w_n) = v_n e^{j2\pi n t / T_0} = v_n e^{jw_n t} \quad (3i-1)$$

The output $v_o(t, w_n)$ is related to the input $v_i(t, w_n)$ by a complex transfer function

$$H(w_n) = |H(w_n)| e^{-j\theta(w_n)} \quad (3i-2)$$

The output is

$$v_o(t, w_n) = H(w_n) v_i(t, w_n) = |H(w_n)| e^{-j\theta(w_n)} v_n e^{jw_n t} = H(w_n) v_n e^{j[w_n t - \theta(w_n)]} \quad (3i-3)$$

The physical input ($v_{ip}(t)$) is the sum of the spectral component and its complex conjugate. i.e.

$$v_{ip}(t, w_n) = v_n e^{jw_n t} + v_n e^{-jw_n t} = v_n e^{jw_n t} + v_n^* e^{-jw_n t} = 2 \text{Re}(v_n e^{jw_n t}) \quad (3i-4)$$

The corresponding physical output is

$$v_{op}(t, w_n) = H(w_n) v_n e^{jw_n t} + H(-w_n) v_n^* e^{-jw_n t} \quad (3i-5)$$

Since the output is +ve and real the two terms in (3i-5) must be complex conjugate. Hence $H(w_n) = H^*(-w_n)$ so $|H(w_n)| = |H(-w_n)|$ and $\theta(w_n) = - \theta(-w_n)$, i.e. $|H(w_n)|$ is an even function and $\theta(w_n)$ is an odd function.

Hence the output of the system can be expressed as
\[ v_0(t) = \sum_{n=-\infty}^{\infty} H(w_n) v_n e^{i2\pi mt/T_0} = H(0)C_0 + \sum_{n=1}^{\infty} H(w_n) C_n \cos \left( \frac{2\pi nt}{T_0} - \phi_n - \theta(w_n) \right) \]

(ii) Normalized Power in a Fourier Expansion:

Consider two terms of the Fourier series expansion (Fundamental and the first harmonics)

\[ v'(t) = C_1 \cos \left( \frac{2\pi t}{T_0} - \phi_1 \right) + C_2 \cos \left( \frac{4\pi t}{T_0} - \phi_2 \right) \]

The normalized power \( S' \) of \( v'(t) \) is

\[ S' = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} [v'(t)]^2 dt = \frac{C_1^2}{2} + \frac{C_2^2}{2} \]

By extension the normalized power associated with the entire Fourier series is

\[ S = C_0^2 + \sum_{n=1}^{\infty} C_n^2 = a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \]

N.B: The power and normalized power are associated with the real waveforms not with the complex waveforms.

For exponential Fourier series the normalized power is due to the product terms

\[ v_n e^{i2\pi mt/T_0} v_{-n} e^{-i2\pi mt/T_0} = v_n v_{-n} = v_n v_n^* \]

Total normalized power is \( S = \sum_{n=-\infty}^{\infty} v_n v_n^* \)

In complex representation, the power associated with a particular frequency \( nf_0 = n/T_0 \) is not associated with the spectral component at \( nf_0 \) and \(-nf_0\), rather the combination of the spectral component. Thus the power is

\[ v_n v_n^* + v_{-n} v_{-n}^* = 2 v_n v_n^* \]
(iii) **Power Spectral Density (PSD):**

The sum $S(f)$ of the normalized power in all spectral components from $f=\pm \infty$ to $\infty$

Normalized power $dS(f)$ at the frequency $f$ in a range $df$ is

$$dS(f) = \frac{dS(f)}{df} \, df$$

$dS(f)/df$ is called the normalized power spectral density $G(f)$.

The power in the range $df$ at $f$ is $G(f)df$. The power in the range $f_1-f_2$ is

$$S(f_1 \leq f \leq f_2) = \int_{f_1}^{f_2} G(f)df$$

And power in the range $-f_2$ to $-f_1$ is

$$S(-f_2 \leq f \leq -f_1) = \int_{-f_2}^{-f_1} G(f)df$$
The quantities in above two equations have no physical significance but the total powers in the real frequency range \( f_1 \leq f \leq f_2 \) have physical significance, and the power is given as

\[
S(|f_1| \leq f \leq |f_2|) = \int_{-f_1}^{f_2} G(f) df + \int_{f_1}^{f_2} G(f) df
\]

To find the power spectral density, differentiate \( S(f) \). But in between the harmonics \( G(f) = 0 \). So at harmonics \( G(f) \) gives an impulse of strength equal to the jump in \( S(f) \). Hence

\[
G(f) = \sum_{n=-\infty}^{\infty} |v_n|^2 \delta(f - nf_0)
\]

(iv) **Effect of Transfer function on PSD:**

Let \( v_i(t) \) is the input to a filter having psd \( G_i(f) \). If \( v_{in} \) is the spectral amplitude of the input signal then

\[
G_i(f) = \sum_{n=-\infty}^{\infty} |v_{in}|^2 \delta(f - nf_0)
\]

Where \( v_{in} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v_i(t)e^{-j2\pi nT_0} dt \)

Let the output is \( v_o \) having spectral amplitude \( v_{on} \), then the corresponding psd is

\[
G_0(f) = \sum_{n=-\infty}^{\infty} |v_{on}|^2 \delta(f - nf_0)
\]

\[
v_{on} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} v_o(t)e^{-j2\pi nT_0} dt
\]

And

If \( H(f) \) is the transfer function of the filter then the input and output spectral amplitudes are related as \( v_{on} = H(f)v_{in} \). Hence \( |v_{on}|^2 = |H(f)|^2 |v_{in}|^2 \)

Substituting in the equation for \( G_0(f) \) above we have \( G_0(f) = G_i(f)|H(f)|^2 \).

**Assignments:**

1. Find the Fourier series expansion for the following wave forms

   (i)
2. Determine the Fourier expansion for the following signals
a. \( x(t) = e^{-n} \) for \( n \leq t \leq n + 1 \)
b. \( x(t) = \cos t + \cos 2.5t \)
c. \( x(t) = |\cos 2\pi f_0 t| \) (full wave rectifier output)
d. \( x(t) = \cos 2\pi f_0 t + |\cos 2\pi f_0 t| \) (half wave rectifier output)
3. Show that for real \( x(t) \)
\[ x_e(t) = \frac{a_0}{2} + \sum a_n \cos(2\pi n f_0 t), \quad \text{and} \quad x_o(t) = \sum a_n \sin(2\pi n f_0 t) \]

Where \( x_e(t) \) and \( x_o(t) \) denote the even and odd parts of \( x(t) \)

\[ x_e(t) = \frac{x(t) + x(-t)}{2} \quad \text{and} \quad x_o(t) = \frac{x(t) - x(-t)}{2} \]

4. Let \( x(t) \) and \( y(t) \) be two periodic signals with period \( T_0 \), and \( x_n \) and \( y_n \) denotes the Fourier series coefficients of these two signals. Show that

\[ \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) y^*(t) dt = \sum_{n=-\infty}^{\infty} x_n y_n^* \]

5. Show that for all periodic physical signal that have finite power, the coefficients of the Fourier series expansion \( x_n \) tend to zero as \( n \to \infty \).

6. A periodic triangular waveform \( v(t) \) is defined by

\[ v(t) = \frac{2t}{T} \quad \text{for} \quad -\frac{T}{2} < t < \frac{T}{2} \quad \text{and} \quad v(t \pm T) = v(t) \]

Calculate the fraction of the normalized power of this waveform which is contained in its first three harmonics.

7. Find \( G(f) \) for the following voltages
   a. An impulse train of strength 1 and period \( T \)
   b. A pulse train of amplitude \( A \), duration \( \tau = \frac{1}{A} \), and period \( T \)

8. Plot \( G(f) \) for a voltage source represented by an impulse train of strength 1 and period \( nT \) for \( n = 1, 2, 10 \), infinity. Comment on this limiting result.

9. \( G_i(f) \) is the power spectral density of a square wave voltage of peak-to-peak amplitude 1 and period 1. The square-wave is filtered by a low-pass RC filter with 3dB frequency 1. The output is taken across the capacitor
   a. Calculate \( G_i(f) \)
   b. Find \( G_o(f) \)

10. (a) A symmetrical square-wave of zero mean value, peak-to-peak voltage 1 volt, and period 1 sec is applied to an ideal low-pass filter. The filter has a transfer function \( |H(f)| = 1/2 \) in the frequency range \(-3.5 \leq f \leq 3.5 \) Hz, and \( H(f) = 0 \) elsewhere. Plot the power spectral density of the filter output
    (b) What is the normalized power of the input square wave? What is the normalized power of the filter output?